Joe Holbrook Memorial Math Competition

6th Grade Solutions

October 18, 2020

1. There are 14 choices for a consonant and 10 choices for a vowel, so $14 * 10 = \boxed{140}$.
2. $2.5 \text{ hr} * \dfrac{60 \text{ min}}{1 \text{ hr}} = 150 \text{ min}$; $\dfrac{5 \text{ things}}{1 \text{ min}} * 150 \text{ min} = \boxed{750}$
3. Alex will take $\dfrac{3000}{50} = 60$ minutes, and Yoland will take $\dfrac{3000}{120} = 25$ minutes, so $60 - 25 = \boxed{35}$ minutes.
4. This problem may be solved as a system of equations. Let x denote the number of cows and let y denote the number of chicken. Then $x + y = 15$ and $4x + 2y = 38$. Solving this equation using substitution or elimination will yield that $x = 4$. Interestingly, this problem can also be solved using a "math team trick." Consider having all chicken, then there will be 30 legs. Converting a chicken to a cow will allow 2 more legs. Therefore, to increase the number of legs from 30 to 38, one should add $\dfrac{8}{2} = \boxed{4}$ cows.
5. The least common multiple of 10 $(2\cdot)$, 15 $(3\cdot)$, and 4 $2\cdot$ is $2 \cdot \cdot\cdot = \boxed{60}$
6. The number of four letter words is $500 - (27 + 166 + 92) = 215$, so the probability of one being selected is $\dfrac{215}{500} = \dfrac{43}{100}$. So $43 + 100 = \boxed{143}$
7. To obtain an odd sum, we can have an even number on one die and an odd number on the other die. This is simply half of all possible cases. So, the answer is $\dfrac{1}{2} \implies \boxed{3}$
8. The 1st and 5th digits are always the same, as well as the 2nd and 4th digits. There are 9 choices for the 1st/5th digit (1, 2, ... , 9), 10 choices for the 2nd/4th digit (0, 1, ... , 9), and 10 choices for the 3rd digit. $9 \cdot 10 \cdot 10 = \boxed{900}$.
9. Working together it will take them $\dfrac{1}{1/3 + 1/6} = 2$ hours. Converting this to minutes, we see that it takes them $\boxed{120}$ minutes to get rid of all the pigs working together.
10. Let the number of pages Autumn had in the beginning be x pages. She throws a third of these pages out the window, leaving her with $x - \dfrac{1}{3}x = \dfrac{2}{3}x$. She then gives a fourth of her remaining pages to her dog, so she now has $\dfrac{2}{3}x \cdot \dfrac{3}{4} = \dfrac{1}{2}x$. Next she gives a fifth of these to Will, so she has $\dfrac{1}{2}x \cdot \dfrac{4}{5} = \dfrac{2}{5}x$ pages left. We know that at this point she had 12 pages left, so $\dfrac{2}{5}x = 12$. Solving for x, we can find that she originally had $x = \dfrac{5}{2} \cdot 12 = \boxed{30}$ pages of homework.
11. Creating a border extending 3 inches from each side results in a rectangle of dimensions 26 inches by 16 inches. Then we need to subtract the poster's area. $26 \cdot 16 - 20 \cdot 10 = \boxed{216}$.
12. $\dfrac{9 \cdot 12}{2} = 54$ boxes can fit vertically, and $\dfrac{16 \cdot 12}{3} = 64$ boxes can fit horizontally. $54 \cdot 64 = \boxed{3456}$
13. Converting feet to inches, we can express our desired scenario as $48 + 2x > 64 - 0.2x$. Solving for x, we get $x > \dfrac{16}{2.2}$. The smallest integer value for x is thus $\boxed{8}$.
14. We just need to subtract the volume of the flesh from the volume of the whole watermelon. The volume of a sphere is $\dfrac{4}{3}\pi r^3$. The radius of the watermelon is $\dfrac{12}{2} = 6$, while the radius of the flesh is $6 - 1 = 5$.

$$\frac{4}{3}\pi(6^3) - \frac{4}{3}\pi(5^3) = \frac{364\pi}{3}$$

$x = \boxed{367}$.

15. A cone has one-third the volume of a cylinder when they have the same radius and height. The cone has half the height of the cylinder, so it has half the volume. Therefore, the cone has $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ the volume of the cylinder. The ratio is $\frac{1}{6}$, so the answer is $1 + 6 = \boxed{7}$.

16. We know that all prime numbers other than 2 are odd. We observe that every number in the set is odd. For the sum of two integers to be odd, one integer must be odd and one must be even. Thus one of the numbers we add to get a number in the set must be even, and the only even prime number is 2. The other number must end with a 5, since 7-2=5. Since any number that ends with a 5 is divisible by 5, there is no prime number other than 5 that ends with 5. Thus the only number in the set that can be written as the sum of two primes is $\boxed{7}$.

17. 10^{300} has a ones digit of 0. We see that

$$7^1 \equiv 7 \pmod{10}$$
$$7^2 \equiv 9 \pmod{10}$$
$$7^3 \equiv 3 \pmod{10}$$
$$7^4 \equiv 1 \pmod{10}$$

$78 \equiv 2 \pmod 4$ so 7^{78} has a ones digit of 9. We also see that

$$3^1 \equiv 3 \pmod{10}$$
$$3^2 \equiv 9 \pmod{10}$$
$$3^3 \equiv 7 \pmod{10}$$
$$3^4 \equiv 1 \pmod{10}$$

$33 \equiv 1 \pmod 4$ so 3^{33} has a ones digit of 3. $0 + 9 + 3 = 2$ so the ones digit of $10^{300} + 7^{78} + 3^{33}$ is $\boxed{2}$.

18. If n is divisible by 5, it must end with a 0 or 5. We see what happens if the hundreds digit is a 9. If the units digit is a 0, and the middle digit is x, then $9 + x + 0$ must be divisible by 6. This happens when $x = 3, 9$. Of these, 990 is the biggest integer we can make. If the units digit is a 5, and the middle digit is y, then $9 + y + 5$ must be divisible by 6. This happens when $y = 4$, making 945. $990 > 945$, so our answer is $\boxed{990}$.

19. If we graph the square the and inequality, this clearly becomes an area problem. We see that the area in the square but above the line $y = -x + 3$ is a triangle with legs of length of 1 each. So it's area is $\frac{1}{2}$. So the area under the line but contained within the square has an area of $\frac{7}{2}$. This over the total area of the square, 4, gives us $\frac{7}{8}$, so the answer will be the product of the numerator and denominator, which is $\boxed{56}$

20. First, note that if there exists a solution (a, b) then (a+5k, b-2k) would work where k is an integer. We can easily see that the largest possible value for b is 18, which creates the pair $(5, 18)$. Now, using our construction, we can also make $(10, 16), (15, 14), \ldots, (45, 2)$. In essence, we are counting the number of even numbers from 2 to 18 inclusive (look at b term). Thus, there are $\boxed{9}$ such ordered pairs.

21. Multiply all of them together and take the square root. This gives $\sqrt{1680} = 4\sqrt{105} \implies \boxed{109}$

22. He can drop 1 ball in 5 ways, 2 in $5 \cdot 4 = 20$ ways, etc to 4 balls in $5 \cdot 4 \cdot 3 \cdot 2 = 120$. Adding these give $\boxed{205}$

23. Complementary counting is very strong here. There are $\binom{6}{3} = 20$ ways to get to $(3, 3)$ without any restrictions. There are $\binom{2}{1}\binom{4}{2} = 12$ ways to get to $(3, 3)$ passing $(1, 1)$. Subtracting these, we get $\boxed{8}$. Furthermore, we can solve the problem this way, if one does not go to $(1, 1)$, one must end up at $(2, 0)$ or $(0, 2)$. At each of these places there are $\binom{4}{1} = 4$ ways to get to $(3, 3)$ for a total of $\boxed{8}$. This can also be brute forced for beginner students.

24. Letting the numbers be a, b, c, d we have that $a+b+c=3, a+b+d=4, a+c+d=5, b+c+d=6$. Summing these gives that $3(a+b+c+d)=18 \implies a+b+c+d=6$. Subtracting each of the other four equations from this equation, we find that the four numbers are $3, 2, 1, 0$. Therefore, $9+4+1=\boxed{14}$.

25. We first find the x values at which the two equations intersect. When $x \geq 0$, $y=|x|=x$. Plugging this into the second equation to eliminate y, we get $x=-x^2+6 \Rightarrow x^2+x-6=0$. Factoring the left hand side gives $(x+3)(x-2)=0$, so $x=2$ (since we stated the $x \geq 0$). By symmetry, the x value of the left intersection is $x=-2$. So we just have to consider the integer x values between these -2 and 2 and find all integer y values so that the coordinate (x, y) is inside the region.

    For $x=-2$, there is 1 point that works: $(-2, 2)$. For $x=-1$, the value of $y=|x|$ is $y=1$ and the value of $y=-x^2+6$ is $y=5$, so all y values between 1 and 5 inclusive work, for a total of 5 points. For $x=0$, the value of $y=|x|$ is $y=0$ and the value of $y=-x^2+6$ is $y=6$, so all y values between 0 and 6 inclusive work, for a total of 7 points. By symmetry, when $x=1$, there are 5 points that work, and when $x=2$, there is 1 point that works.

    In total, there are $1+5+7+5+1=\boxed{19}$ lattice points in the region or on the boundary.

26. Notice that consecutive terms can be factored into [(i+1) - (i-1)]*i, so the sum collapses into a sum of consecutive integers times 4. Continuing this arithmetic yields a sum of $\boxed{5100}$.

27. Let's say the expected number of mango pieces Alicia will eat is E. If the piece she eats is sour, then she will only be eating that piece. If the piece she eats is sweet, then she will eat $\frac{2}{5} \cdot E+1$ pieces. This yields the equation, $\frac{3}{5} \cdot 1+\frac{2}{5} \cdot (E+1)=E$. Solving this equation, we get $E=\frac{5}{3}$ and the answer is $5+3=\boxed{8}$.

28. Letting $x=\sqrt{12-\sqrt{12-\sqrt{12-\sqrt{12-\cdots}}}}$, we have $x=\sqrt{12-x}$. Squaring both sides gives $x^2=12-x$, so $x^2+x-12=0$. Factoring the left side gives $(x-3)(x+4)=0$. Therefore, $x=-4$ or $x=3$. Clearly x must be positive, so we have $x=\boxed{3}$.

29. By Vieta's formulas, a, b, c are the roots of the polynomial

    $$x^3-3x^2+3x-3=0.$$

    Adding 2 to both sides, we can factor this equation as

    $$(x-1)^3=2.$$

    For the real value $x=a$, we have $a-1=\sqrt[3]{2}$, so $a=1+\sqrt[3]{2}$. So the answer is $1+3+2=\boxed{6}$

30. Assume that $x=r$ is a root of this equation. Then if we plug in $x=1-r$, our equation becomes $(1-r)^{45}+(1-(1-r))^{45}=(1-r)^{45}+r^{45}=0$, which is identical to our original equation. Thus, for every root r, $1-r$ is also a root to this equation. We can see that the coefficient of the x^{45} term in the expansion of $(1-x)^{45}$ is -1, so the x^{45} terms cancel out. We thus have a polynomial of degree 44, so there are 44 roots. Since we can see that $\frac{1}{2}$ is not a root, there must be 44 distinct roots and 22 pairs of roots. Each pair of roots adds to 1, so the sum of the roots is $\boxed{22}$.

31. We want to maximize the number of sections we get with each cut. To do this, notice that each cut should intersect all previous cuts, but not at preexisting intersection points. So the nth cut would intersect $n-1$ chords, creating n new sections. We quickly realize that with n cuts, the maximum number of sections is $1+T_n$, where T_n is the nth triangular number, or $1+\frac{n(n+1)}{2}$. Since we have 138 people (including the captain), $1+\frac{n(n+1)}{2} \geq 138$. The smallest value of n is $\boxed{17}$.

32. If all the bags were made in one microwave, it would be a total of $\frac{101 \cdot 102}{2}=5151$ minutes of popping, meaning that if the action were spread over two microwaves, it should take at least the ceiling of $\frac{5151}{2}$ minutes, or 2576. This is also attainable, as making the bags $1, 2, 3, \cdot, 69, 70$ as well as 91 in the same microwave will take 2576 minutes, so the answer is $\boxed{2576}$.

33. We start by realizing that the number of coins in a pouch is equal to the sum of the factors of the bag's number. For instance, the sixth bag has 12 coins because People 1, 2, 3, and 6 dropped 1, 2, 3, and 6 coins, respectively. Therefore, we must determine which numbers from 1 through 100 have an odd sum of factors. The sum of factors for any number is calculated by taking each different prime factor and adding together all its powers up to the one that appears in the prime factorization, and then multiply all these sums together. For instance, Pouch 24 would have $(1+2+4+8)(1+3)=60$ coins. To generate an odd product, we must have only odd factors. Powers of 2 always generate odd factors since it is always $1+(powers\ of\ 2)$. Powers of other primes generate odd factors only when the power is even. For instance 3^2 generates $1+3+9=13$ but 3^3 generates $1+3+9+27=40$. With this in mind, all pouches with an odd number of coins will have powers of 2 and/or odd primes raised to even powers in their prime factorization. For powers of 2 we have $2, 2^2, 2^3, 2^4, 2^5, 2^6$, for powers of 3 we have $3^2, 3^4$, for powers of 5 we have 5^2, and for powers of 7 we have 7^2. Multiplying these numbers together yields 16 different numbers less than 100: 2, 4, 8, 16, 32, 64, 9, 18, 36, 72, 81, 25, 50, 100, 49, and 98. The final pouch is Pouch 1, yielding a total of $\boxed{17}$ pouches.

34. Since $AB=AC=AD$, we can draw a semicircle with A as the center and AB as the radius. Let B' be the point on the circle that's on the diameter with A and B. Then $BCDB'$ is an isosceles trapezoid, meaning $B'D=4$. $B'DB$ is a right triangle because one of its sides is the diameter of the circle and all three vertices are on the circle, so $BD=\sqrt{B'B^2-DB'^2}=\sqrt{10^2-4^2}=2\sqrt{21}$ and $a+b=\boxed{23}$.

35. Let the number of blue socks be x. Therefore, we have $x+10$ red socks and $2x+10$ in total. We also know that the probability of Bobette getting a pair of same-colored socks is $\frac{1}{2}$. The number of ways to get a pair of red socks is $(x+10)(x+9)$ and the number of ways to get a pair of blue socks is $(x)(x-1)$. Furthermore, the total number of ways to choose two socks is $(2x+10)(2x+9)$. We can now write

$$\frac{(x+10)(x+9)+(x)(x-1)}{(2x+10)(2x+9)}=\frac{1}{2}.$$

Solving for x we get 45. Therefore, Bobette has $\boxed{45}$ blue socks.

36. There are two parts of the perimeter - sides AD and DC, and sides AB and BC. If E and F are on different parts, the segments will clearly not intersect. If the points are on the same part, one point will be closer to A than C, assuming the points are distinct. The segments will intersect if and only F is closer to A. There is a 1/2 probability that the points are on the same part and a 1/2 probability that F is closer to A. Thus the probability the segments intersect is $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$. The answer is $\boxed{5}$.

37. On the last lily pad, the amount of terms it will take to reach the last lily pad is $E_0=0$. For any other lily pad, we have $E_n=1+\frac{1}{n}(E_0+E_1+...+E_{n-1})$. It is possible to simplify this into an explicit formula for E_n but that is not necessary, as calculating the $E0...4$ by hand is easy: $E_0=0, E_1=1, E_2=\frac{3}{2}, E_3=\frac{11}{6}$, and finally $x=E_4=\frac{25}{12}$, yielding the answer $\boxed{37}$.

38. We can establish a 1-1 correspondence between the number of ways for Justin to win and the number of ways to choose 4 rows and 4 columns in a six by six grid. To see why this is, we can consider a simpler case. Suppose that instead of choosing 4 spots for 4 non distinct balls, we are choosing 2 spots for 2 non distinct balls. Given any combination of 2 rows and 2 columns that intersect at 4 squares, there exists a unique orientation of 2 balls in those 4 squares satisfying the condition that no balls lies on the right of and above another ball and that there are no balls in the same row and column. Therefore, the number of ways for Justin to win is $\binom{6}{4}\cdot\binom{6}{4}$, and the number of ways to choose 4 balls out of $6^2=36$ spots is $\binom{36}{4}$. Hence, the desired probability is $\frac{\binom{6}{4}\cdot\binom{6}{4}}{\binom{36}{4}}$, which simplifies to $\frac{5}{7\cdot 17\cdot 11}$. $m=5, n=7\cdot 17\cdot 11 \implies m+\frac{n}{11}=5+\frac{7\cdot 17\cdot 11}{11}=5+119=\boxed{124}$.

39. Consider the dilation centered at C that takes ω_A to ω_B. This maps D to some point D' on ω_B. Therefore, $CD'=\frac{5}{3}CD$, and it follows that because $\triangle CD'E$ and $\triangle CDE$ share the same height, the ratio of the area of $\triangle CD'E$ to the area of $\triangle CDE$ is the ratio of their bases, namely, $\frac{5}{3}$. Therefore, if we can maximize the area of $\triangle CD'E$, then we can just multiply this number by $\frac{3}{5}$ to get the desired area. The maximum

area of $\triangle CD'E$ occurs when it is equilateral, and this gives us an area of $\frac{75\sqrt{3}}{4}$. Therefore, the maximum area of $\triangle CDE$ is $\frac{45\sqrt{3}}{4}$, and the answer is $45 + 3 + 4 = \boxed{52}$.

40. "Let C be the total coastline, and let A be the total area. Then the following two equations hold:

$$C = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 2\pi \left(\frac{1}{2^{m+n+1}}\right)$$

$$A = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \pi \left(\frac{1}{2^{m+n+1}}\right)^2$$

We can rewrite the expression for C as:

$$C = \pi\left(\sum_{m=0}^{\infty}\frac{1}{2^m}\right)\left(\sum_{n=0}^{\infty}\frac{1}{2^n}\right) = \pi \cdot 2 \cdot 2 = 4\pi$$

As for A, we get that:

$$A = \frac{\pi}{4}\left(\sum_{m=0}^{\infty}\frac{1}{4^m}\right)\left(\sum_{n=0}^{\infty}\frac{1}{4^n}\right) = \frac{\pi}{4}\cdot\frac{4}{3}\cdot\frac{4}{3} = \frac{4\pi}{9}$$

We can now compute that $\frac{C}{A} = \boxed{9}$."